

Moments of Progressive Type-II Right Censored Order Statistics from Lindley Distribution

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Abstract

In this article, we establish some new recurrence relations between moments of progressively Type-II right censored order statistics from Lindley distribution. For practical application, these relations in a simple recursive manner may be used to compute all the single and product moments, and hence all means, variances and covariances of progressively Type-II right censored order statistics from Lindley distribution for all sample sizes n , effective sample sizes m , and all progressive censoring schemes $(R_1, R_2, \dots, R_m)$, $m \leq n$. In the last section, some deduction and particular cases are also discussed.

Keywords

Progressive Type-II Right Censored Order Statistics; Order Statistics; Single Moments; Product Moments; Recurrence Relations; Lindley Distribution

Introduction

There are many practical situations in life-testing and reliability experiments in which items are lost or eliminated from experimentation before failure. Life tests and reliability experiments are usually conducted to obtain information regarding the quality of a product. According to Hamada *et al.* (2008) "One of the features of reliability data is the presence of censoring. Lifetime data are censored when the exact failure time for a specific item is unknown". One can also refer to Lawless (1982) for more about censoring. There are several types of censoring schemes used in lifetime analysis. Here a very useful censoring scheme called progressive Type-II censored sampling scheme is considered, which allows the experimenter to save time and cost of the life-testing experiment. Progressive Type-II censoring was introduced by Cohen (1963), under which n independent and identical items are put on a life-test with continuous identically distributed failure times. Suppose a

censoring schemes R_i 's is prefixed integers such that at the time of first failure, R_1 surviving items are removed from the experiment at random from the remaining $n-1$ surviving items. At the time of the second observed failure, R_2 surviving items are removed from the experiment at random from the remaining $n-2-R_1$ surviving items. The process continues until the m -th failure time at which all the remaining $R_m = n - m - \sum_{t=1}^{m-1} R_t$ (with $\sum_{i=a}^b R_i = 0, b < a$) surviving items are removed from the experiment at random. We shall denote the m ordered observed failure times by $X_{1:m:n}^{(R_1, R_2, \dots, R_m)}, \dots, X_{m:m:n}^{(R_1, R_2, \dots, R_m)}$ and call them the progressive Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, \dots, R_m)$, $m \leq n$. If the failure times of the n items originally on test are from a continuous population with distribution function (df) F and probability density function (pdf) f , then the joint pdf of $X_{1:m:n}^{(R_1, R_2, \dots, R_m)}, \dots, X_{m:m:n}^{(R_1, R_2, \dots, R_m)}$ is given by [Balakrishnan and Sandhu (1995), Balakrishnan (2007)]

$$f_{X_{1:m:n}, \dots, X_{m:m:n}}(x_1, x_2, \dots, x_m) = A(n, m-1) \prod_{i=1}^m f(x_i) [1-F(x_i)]^{R_i},$$

$$0 < x_1 < x_2 < \dots < x_m < \infty, \quad (1.1)$$

where,

$$A(n, m-1) = n(n-R_1-1) \dots (n-R_1-R_2-\dots-R_{m-1}-m+1),$$

with $A(n, 0) = n$.

Progressive censoring and associated inferential procedures have been extensively studied in the literature for a number of distributions by several authors. Cohen (1963, 1966, 1975, 1976 and 1991), Mann (1969, 1971), Cohen and Whitten (1988), Viveros and Balakrishnan (1994), Balakrishnan and Sandhu

(1995), Aggrawala and Balakrishnan (1996) and Balakrishnan and Aggrawala (2000) have derived recurrence relations for single and product moments of progressively Type-II right censored order statistics from exponential, Pareto and power function distribution and their truncated forms. Saran and Pushkarna (2001) have obtained several recurrence relations for the single and product moments of progressively Type-II right censored order statistics from doubly truncated Burr distribution. Mahmoud *et al.* (2006) derived some new recurrence relations for single and product moments of progressively Type-II right censored order statistics from linear exponential distribution and also obtain maximum likelihood estimators (MLEs) of the location and scale parameters of the linear exponential distribution. Balakrishnan *et al.* (2011) and Balakrishnan and Saleh (2011, 2012a, b) have established several recurrence relations for single and product moments of progressively Type-II right censored order statistics from logistic, half-logistic, generalized half logistic and log-logistic distributions and the moments so determined are then utilized in inferential method to derive best linear unbiased estimators of the scale and location-scale parameters. Saran and Pande (2012) also obtained some recurrence relations for progressively Type-II right censored order statistics from half-logistic distribution.

The exponential distribution is a basic physical model in reliability theory and survival analysis and due to the popularity of the exponential distribution in statistics and many applied areas; Lindley distribution has drawn little attention in the statistical literature. Lindley distribution was first introduced by Lindley (1958) in connection with the Fiducial distribution and Bayes theorem. The Lindley distribution is important for studying stress-strength reliability modelling [Hussain (2006)]. Ghitany *et al.* (2008) was the first who studied the statistical properties of the said distribution and it was shown that in many ways the Lindley distribution is a better model than that based on the exponential distribution.

Let $X_{1:m:n}^{(R_1, \dots, R_m)} < X_{2:m:n}^{(R_1, \dots, R_m)} < \dots < X_{m:m:n}^{(R_1, \dots, R_m)}$ be the m ordered observed failure times in a sample of size n under progressive Type-II right censoring scheme $(R_1, R_2, \dots, R_m)$ from the Lindley distribution with pdf ,

$$f(x) = \frac{\lambda^2}{1+\lambda} (1+x) e^{-\lambda x}, x > 0, \lambda > 0, \quad (1.2)$$

and df

$$F(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x}, x > 0, \lambda > 0. \quad (1.3)$$

It may be noted that from (1.2) and (1.3),

$$(1+\lambda+\lambda x) f(x) = \lambda^2 \{[1-F(x)] + x[1-F(x)]\}. \quad (1.4)$$

The relation in (1.4) is the “characterizing differential equation” for the distribution in (1.2).

Main Results

For any continuous distribution, we shall denote the i -th single moment of the progressively Type-II right censored order statistics in view of (1.1) as

$$\begin{aligned} \mu_{r:m:n}^{(R_1, R_2, \dots, R_m)^{(i)}} &= E[X_{r:m:n}^{(R_1, R_2, \dots, R_m)}]^i \\ &= A(n, m-1) \iint \dots \int_{0 < x_1 < \dots < x_m < \infty} x_r^i f(x_1) [1-F(x_1)]^{R_1} \\ &\quad \times f(x_2) [1-F(x_2)]^{R_2} \dots f(x_m) [1-F(x_m)]^{R_m} dx_1 \dots dx_m, \end{aligned} \quad (2.1)$$

and the i -th and j -th product moments as

$$\begin{aligned} \mu_{r,s:m:n}^{(R_1, R_2, \dots, R_m)^{(i,j)}} &= E[X_{r:m:n}^{(R_1, R_2, \dots, R_m)^{(i)}} X_{s:m:n}^{(R_1, R_2, \dots, R_m)^{(j)}}] \\ &= A(n, m-1) \iint \dots \int_{0 < x_1 < \dots < x_m < \infty} x_r^i x_s^j f(x_1) [1-F(x_1)]^{R_1} \\ &\quad \times f(x_2) [1-F(x_2)]^{R_2} \dots f(x_m) [1-F(x_m)]^{R_m} dx_1 \dots dx_m, \end{aligned} \quad (2.2)$$

where $A(n, m-1)$ is defined as above.

Theorem 2.1: For $s \geq 2$, $3 \leq m \leq n$, and $i, j \geq 0$,

$$\begin{aligned} \mu_{1,s:m:n}^{(R_1, \dots, R_m)^{(i+2,j)}} &= \frac{(i+2)}{\lambda(R_1+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{1,s:m:n}^{(R_1, \dots, R_m)^{(i,j)}} + \left\{ 1 - \frac{\lambda(R_1+1)}{(i+1)} \right\} \mu_{1,s:m:n}^{(R_1, \dots, R_m)^{(i+1,j)}} \right] \\ &\quad - \frac{(n-R_1-1)}{(R_1+1)} \left[\mu_{1,s-1;m-1:n}^{(R_1+R_2+1, R_3, \dots, R_m)^{(i+2,j)}} + \left\{ \frac{i+2}{i+1} \right\} \mu_{1,s-1;m-1:n}^{(R_1+R_2+1, R_3, \dots, R_m)^{(i+1,j)}} \right] \end{aligned} \quad (2.3)$$

Proof. From (2.2), we have

$$\begin{aligned} (1+\lambda) \mu_{1,s:m:n}^{(R_1, \dots, R_m)^{(i,j)}} + \lambda \mu_{1,s:m:n}^{(R_1, \dots, R_m)^{(i+1,j)}} &= A(n, m-1) \iint \dots \int_{0 < x_2 < \dots < x_m < \infty} x_s^j Z(x_2) f(x_2) [1-F(x_2)]^{R_2} \\ &\quad \times \dots \times f(x_m) [1-F(x_m)]^{R_m} dx_2 \dots dx_m, \end{aligned} \quad (2.4)$$

where,

$$Z(x_2) = \int_0^{x_2} x_1^i (1+\lambda+\lambda x_1) f(x_1) [1-F(x_1)]^{R_1} dx_1. \quad (2.5)$$

On using relation (1.4) in (2.5) and integrating by parts, we get

$$\begin{aligned} Z(x_2) &= \lambda^2 \left[\frac{1}{(i+1)} \left\{ x_2^{i+1} [1-F(x_2)]^{R_1+1} + (R_1+1) \int_0^{x_2} x_1^{i+1} [1-F(x_1)]^{R_1} f(x_1) dx_1 \right\} \right. \\ &\quad \left. + \frac{1}{(i+2)} \left\{ x_2^{i+2} [1-F(x_2)]^{R_1+1} + (R_1+1) \int_0^{x_2} x_1^{i+2} [1-F(x_1)]^{R_1} f(x_1) dx_1 \right\} \right]. \end{aligned} \quad (2.6)$$

Now by substituting the resultant expression of $Z(x_2)$ from (2.6) in (2.4) and simplifying, we get the required result.

Corollary 2.1: For $1 \leq m \leq n$ and $i \geq 0$,

$$\begin{aligned} \mu_{1,m:n}^{(R_1, \dots, R_m)^{(i+2)}} &= \frac{(i+2)}{\lambda(R_1+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{1,m:n}^{(R_1, \dots, R_m)^{(i)}} + \left\{ 1 - \frac{\lambda(R_1+1)}{(i+1)} \right\} \mu_{1,m:n}^{(R_1, \dots, R_m)^{(i+1)}} \right] \end{aligned}$$

$$-\frac{(n-R_1-1)}{(R_1+1)} \left[\mu_{1:m-1:n}^{(R_1+R_2+1, R_3, \dots, R_m)^{(i+2)}} + \left\{ \frac{i+2}{i+1} \right\} \mu_{1:m-1:n}^{(R_1+R_2+1, R_3, \dots, R_m)^{(i+1)}} \right] \quad (2.7)$$

and subsequently for $m=1$,

$$\begin{aligned} & \mu_{1:1:n}^{(n-1)^{(i+2)}} \\ &= \frac{(i+2)}{n\lambda} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{1:1:n}^{(n-1)^{(i)}} + \left\{ 1 - \frac{n\lambda}{(i+1)} \right\} \mu_{1:1:n}^{(n-1)^{(i+1)}} \right]. \quad (2.8) \end{aligned}$$

Proof. Relation (2.7) is the simple consequence of (2.3) and can be established by putting $j=0$ in (2.3). Further, (2.8) can be seen by putting $m=1$ in (2.7) after noting that $R_1 = n-1$.

Corollary 2.2: For $2 \leq m \leq n$ and $i, j \geq 0$,

$$\begin{aligned} & \mu_{1,2:m:n}^{(R_1, \dots, R_m)^{(i+2,j)}} \\ &= \frac{(i+2)}{\lambda(R_1+1)} \left[\frac{(1+\lambda)}{\lambda} \mu_{1,2:m:n}^{(R_1, \dots, R_m)^{(i,j)}} + \left\{ 1 - \frac{\lambda(R_1+1)}{(i+1)} \right\} \mu_{1,2:m:n}^{(R_1, \dots, R_m)^{(i+1,j)}} \right] \\ & - \frac{(n-R_1-1)}{(R_1+1)} \left[\mu_{1:m-1:n}^{(R_1+R_2+1, R_3, \dots, R_m)^{(i+j+2)}} + \left\{ \frac{i+2}{i+1} \right\} \mu_{1:m-1:n}^{(R_1+R_2+1, R_3, \dots, R_m)^{(i+j+1)}} \right]. \quad (2.9) \end{aligned}$$

Proof. (2.9) can be proved by putting $s=2$ in (2.3).

Theorem 2.2: For $1 \leq r < s \leq m-1$, $m \leq n$ and $i, j \geq 0$,

$$\begin{aligned} & \mu_{r,s:m:n}^{(R_1, \dots, R_m)^{(i,j+2)}} \\ &= \frac{(j+2)}{\lambda(R_s+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{r,s:m:n}^{(R_1, \dots, R_m)^{(i,j)}} + \left\{ 1 - \frac{\lambda(R_s+1)}{(j+1)} \right\} \mu_{r,s:m:n}^{(R_1, \dots, R_m)^{(i,j+1)}} \right] \\ & - \frac{(n-R_1-\dots-R_s-s)}{(R_s+1)} \left[\mu_{r,s:m-1:n}^{(R_1, \dots, R_{s-1}, R_s+R_{s+1}+1, \dots, R_m)^{(i,j+2)}} \right. \\ & + \left. \left\{ \frac{j+2}{j+1} \right\} \mu_{r,s:m-1:n}^{(R_1, \dots, R_{s-1}, R_s+R_{s+1}+1, \dots, R_m)^{(i,j+1)}} \right] \\ & + \frac{(n-R_1-\dots-R_{s-1}-s+1)}{(R_s+1)} \left[\mu_{r,s-1:m-1:n}^{(R_1, \dots, R_{s-2}, R_{s-1}+R_s+1, \dots, R_m)^{(i,j+2)}} \right. \\ & + \left. \left\{ \frac{j+2}{j+1} \right\} \mu_{r,s-1:m-1:n}^{(R_1, \dots, R_{s-2}, R_{s-1}+R_s+1, \dots, R_m)^{(i,j+1)}} \right]. \quad (2.10) \end{aligned}$$

Proof. From (2.2), we get

$$\begin{aligned} & (1+\lambda) \mu_{r,s:m:n}^{(R_1, \dots, R_m)^{(i,j)}} + \lambda \mu_{r,s:m:n}^{(R_1, \dots, R_m)^{(i,j+1)}} \\ &= (1+\lambda) E[X_{r:m:n}^{(R_1, \dots, R_m)^{(i)}} X_{s:m:n}^{(R_1, \dots, R_m)^{(j)}}] + \lambda E[X_{r:m:n}^{(R_1, \dots, R_m)^{(i)}} X_{s:m:n}^{(R_1, \dots, R_m)^{(j+1)}}] \\ &= A(n, m-1) \int \dots \int_{0 < x_1 < \dots < x_{s-1} < x_{s+1} < \dots < x_m < \infty} x_r^i Z(x_{s-1}, x_{s+1}) f(x_1) [1-F(x_1)]^{R_1} \\ & \times \dots \times f(x_{s-1}) [1-F(x_{s-1})]^{R_{s-1}} f(x_{s+1}) [1-F(x_{s+1})]^{R_{s+1}} \\ & \times \dots \times f(x_m) [1-F(x_m)]^{R_m} dx_1 \dots dx_{s-1} dx_{s+1} \dots dx_m, \quad (2.11) \end{aligned}$$

where

$$Z(x_{s-1}, x_{s+1}) = \int_{x_{s-1}}^{x_{s+1}} x_s^j (1+\lambda + \lambda x_s) f(x_s) [1-F(x_s)]^{R_s} dx_s. \quad (2.12)$$

On using relation (1.4) in (2.12) and integrating by parts, we get

$$\begin{aligned} & Z(x_{s-1}, x_{s+1}) \\ &= \lambda^2 \left[\frac{1}{(j+1)} \left(x_{s+1}^{j+1} [1-F(x_{s+1})]^{R_s+1} - x_{s-1}^{j+1} [1-F(x_{s-1})]^{R_s+1} \right) \right. \end{aligned}$$

$$\begin{aligned} & + (R_s+1) \int_{x_{s-1}}^{x_{s+1}} x_s^{j+1} [1-F(x_s)]^{R_s} f(x_s) dx_s \Big) \\ & + \frac{1}{(j+2)} \left(x_{s+1}^{j+2} [1-F(x_{s+1})]^{R_s+1} - x_{s-1}^{j+2} [1-F(x_{s-1})]^{R_s+1} \right. \\ & + (R_s+1) \int_{x_{s-1}}^{x_{s+1}} x_s^{j+2} [1-F(x_s)]^{R_s} f(x_s) dx_s \Big). \quad (2.13) \end{aligned}$$

Now upon substituting the resultant expression of $Z(x_{s-1}, x_{s+1})$ from (2.13) in (2.11) and simplifying, yields (2.10).

Corollary 2.3: For $2 \leq r \leq m-1$, $m \leq n$ and $i \geq 0$,

$$\begin{aligned} & \mu_{r:m:n}^{(R_1, \dots, R_m)^{(i+2)}} \\ &= \frac{(i+2)}{\lambda(R_r+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{r:m:n}^{(R_1, \dots, R_m)^{(i)}} + \left\{ 1 - \frac{\lambda(R_r+1)}{(i+1)} \right\} \mu_{r:m:n}^{(R_1, \dots, R_m)^{(i+1)}} \right] \\ & - \frac{(n-R_1-\dots-R_r-r)}{(R_r+1)} \left[\mu_{r:m-1:n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, R_{r+2}, \dots, R_m)^{(i+2)}} \right. \\ & + \left. \left\{ \frac{i+2}{i+1} \right\} \mu_{r:m-1:n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, R_{r+2}, \dots, R_m)^{(i+1)}} \right] \\ & + \frac{(n-R_1-\dots-R_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1:m-1:n}^{(R_1, \dots, R_{r-2}, R_{r-1}+R_r+1, R_{r+1}, \dots, R_m)^{(i+2)}} \right. \\ & + \left. \left\{ \frac{i+2}{i+1} \right\} \mu_{r-1:m-1:n}^{(R_1, \dots, R_{r-2}, R_{r-1}+R_r+1, R_{r+1}, \dots, R_m)^{(i+1)}} \right]. \quad (2.14) \end{aligned}$$

Proof. Relation (2.14) can be established by putting $i=0$ in (2.10) and replacing s by r .

Corollary 2.4: For $2 \leq m \leq n$ and $i \geq 0$,

$$\begin{aligned} & \mu_{m:m:n}^{(R_1, \dots, R_m)^{(i+2)}} \\ &= \frac{(i+2)}{\lambda(R_m+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{m:m:n}^{(R_1, \dots, R_m)^{(i)}} + \left\{ 1 - \frac{\lambda(R_m+1)}{(i+1)} \right\} \mu_{m:m:n}^{(R_1, \dots, R_m)^{(i+1)}} \right] \\ & + \frac{(n-R_1-\dots-R_{m-1}-m+1)}{(R_m+1)} \left[\mu_{m-1:m-1:n}^{(R_1, \dots, R_{m-2}, R_{m-1}+R_m+1)^{(i+2)}} \right. \\ & + \left. \left\{ \frac{i+2}{i+1} \right\} \mu_{m-1:m-1:n}^{(R_1, \dots, R_{m-2}, R_{m-1}+R_m+1)^{(i+1)}} \right]. \quad (2.15) \end{aligned}$$

Proof. (2.15) can be proved by replacing r with m in (2.14) and noting that $\mu_{m:m-1:n}^{(R_1, R_2, \dots, R_m)^{(i)}} = 0$

Corollary 2.5: For $1 \leq r \leq m-2$, $m \leq n$ and $i, j \geq 0$,

$$\begin{aligned} & \mu_{r,r+1:m:n}^{(R_1, \dots, R_m)^{(i,j+2)}} \\ &= \frac{(j+2)}{\lambda(R_{r+1}+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{r,r+1:m:n}^{(R_1, \dots, R_m)^{(i,j)}} + \left\{ 1 - \frac{\lambda(R_{r+1}+1)}{(j+1)} \right\} \mu_{r,r+1:m:n}^{(R_1, \dots, R_m)^{(i,j+1)}} \right] \\ & - \frac{(n-R_1-\dots-R_{r+1}-r-1)}{(R_{r+1}+1)} \left[\mu_{r,r+1:m-1:n}^{(R_1, \dots, R_r, R_{r+1}+R_{r+2}+1, \dots, R_m)^{(i,j+2)}} \right. \\ & + \left. \left\{ \frac{j+2}{j+1} \right\} \mu_{r,r+1:m-1:n}^{(R_1, \dots, R_r, R_{r+1}+R_{r+2}+1, \dots, R_m)^{(i,j+1)}} \right] \\ & + \frac{(n-R_1-\dots-R_r-r)}{(R_{r+1}+1)} \left[\mu_{r:m-1:n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, \dots, R_m)^{(i+j+2)}} \right. \\ & + \left. \left\{ \frac{j+2}{j+1} \right\} \mu_{r:m-1:n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, \dots, R_m)^{(i+j+1)}} \right]. \quad (2.16) \end{aligned}$$

Proof. (2.16) can be proved by putting $s = r + 1$ in (2.10).

Corollary 2.6: For $1 \leq r \leq m-1$, $m \leq n$ and $i, j \geq 0$,

$$\begin{aligned} & \mu_{r,m;n}^{(R_1, \dots, R_m)^{(i,j+2)}} \\ &= \frac{(j+2)}{\lambda(R_m+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{r,m;n}^{(R_1, \dots, R_m)^{(i,j)}} + \left\{ 1 - \frac{\lambda(R_m+1)}{(j+1)} \right\} \mu_{r,m;n}^{(R_1, \dots, R_m)^{(i,j+1)}} \right] \\ &+ \frac{(n-R_1-\dots-R_{m-1}-m+1)}{(R_m+1)} \left[\mu_{r,m-1;n}^{(R_1, \dots, R_{m-2}, R_{m-1}+R_m+1)^{(i,j+2)}} \right. \\ &\left. + \left\{ \frac{j+2}{j+1} \right\} \mu_{r,m-1;n}^{(R_1, \dots, R_{m-2}, R_{m-1}+R_m+1)^{(i,j+1)}} \right]. \end{aligned} \quad (2.17)$$

Proof. Corollary can be proved by putting $s = m$ in (2.10).

Theorem 2.3: For $1 \leq r < s \leq m-1$, $m \leq n$ and $i, j \geq 0$,

$$\begin{aligned} & \mu_{r,s;m;n}^{(R_1, \dots, R_m)^{(i+2,j)}} \\ &= \frac{(i+2)}{\lambda(R_r+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{r,s;m;n}^{(R_1, \dots, R_m)^{(i,j)}} + \left\{ 1 - \frac{\lambda(R_r+1)}{(i+1)} \right\} \mu_{r,s;m;n}^{(R_1, \dots, R_m)^{(i+1,j)}} \right] \\ &- \frac{(n-R_1-\dots-R_r-r)}{(R_r+1)} \left[\mu_{r,s-1;m-1;n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, \dots, R_m)^{(i+2,j)}} \right. \\ &\left. + \left\{ \frac{i+2}{i+1} \right\} \mu_{r,s-1;m-1;n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, \dots, R_m)^{(i+1,j)}} \right] \\ &+ \frac{(n-R_1-\dots-R_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1,s-1;m-1;n}^{(R_1, \dots, R_{r-2}, R_{r-1}+R_r+1, \dots, R_m)^{(i+2,j)}} \right. \\ &\left. + \left\{ \frac{i+2}{i+1} \right\} \mu_{r-1,s-1;m-1;n}^{(R_1, \dots, R_{r-2}, R_{r-1}+R_r+1, \dots, R_m)^{(i+1,j)}} \right]. \end{aligned} \quad (2.18)$$

Proof. From (2.2), we have

$$\begin{aligned} & (1+\lambda) \mu_{r,s;m;n}^{(R_1, \dots, R_m)^{(i,j)}} + \lambda \mu_{r,s;m;n}^{(R_1, \dots, R_m)^{(i+1,j)}} \\ &= (1+\lambda) E[X_{r,m;n}^{(R_1, \dots, R_m)^{(i)}} X_{s,m;n}^{(R_1, \dots, R_m)^{(j)}}] + \lambda E[X_{r,m;n}^{(R_1, \dots, R_m)^{(i+1)}} X_{s,m;n}^{(R_1, \dots, R_m)^{(j)}}] \\ &= A(n, m-1) \int \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_m < \infty} x_s^j Z(x_{r-1}, x_{r+1}) f(x_1) [1-F(x_1)]^{R_1} \\ &\times \dots \times f(x_{r-1}) [1-F(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [1-F(x_{r+1})]^{R_{r+1}} \\ &\times \dots \times f(x_m) [1-F(x_m)]^{R_m} dx_1 \dots dx_{s-1} dx_{s+1} \dots dx_m, \end{aligned} \quad (2.19)$$

where

$$Z(x_{r-1}, x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^i (1+\lambda + \lambda x_r) f(x_r) [1-F(x_r)]^{R_r} dx_r. \quad (2.20)$$

On using relation (1.4) in (2.20) and integrating by parts, we get

$$\begin{aligned} & Z(x_{r-1}, x_{r+1}) \\ &= \lambda^2 \left[\frac{1}{(i+1)} (x_{r+1}^{i+1} [1-F(x_{r+1})]^{R_{r+1}} - x_{r-1}^{i+1} [1-F(x_{r-1})]^{R_{r+1}} \right. \\ &+ (R_r+1) \int_{x_{r-1}}^{x_{r+1}} x_r^{i+1} [1-F(x_r)]^{R_r} f(x_r) dx_r \Big) \\ &+ \frac{1}{(i+2)} (x_{r+1}^{i+2} [1-F(x_{r+1})]^{R_{r+1}} - x_{r-1}^{i+2} [1-F(x_{r-1})]^{R_{r+1}} \\ &+ (R_r+1) \int_{x_{r-1}}^{x_{r+1}} x_r^{i+2} [1-F(x_r)]^{R_r} f(x_r) dx_r \Big). \end{aligned} \quad (2.21)$$

Now by substituting the resultant expression of

$Z(x_{r-1}, x_{r+1})$ from (2.21) in (2.19) and simplifying, leads to (2.18).

Corollary 2.7: For $2 \leq r \leq m-2$, $m \leq n$ and $i, j \geq 0$,

$$\begin{aligned} & \mu_{r,r+1;m;n}^{(R_1, \dots, R_m)^{(i+2,j)}} \\ &= \frac{(i+2)}{\lambda(R_r+1)} \left[\left\{ \frac{1+\lambda}{\lambda} \right\} \mu_{r,r+1;m;n}^{(R_1, \dots, R_m)^{(i,j)}} + \left\{ 1 - \frac{\lambda(R_r+1)}{(i+1)} \right\} \mu_{r,r+1;m;n}^{(R_1, \dots, R_m)^{(i+1,j)}} \right] \\ &- \frac{(n-R_1-\dots-R_r-r)}{(R_r+1)} \left[\mu_{r,m-1;n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, \dots, R_m)^{(i+2,j)}} \right. \\ &\left. + \left\{ \frac{i+2}{i+1} \right\} \mu_{r,m-1;n}^{(R_1, \dots, R_{r-1}, R_r+R_{r+1}+1, \dots, R_m)^{(i+1,j)}} \right] \\ &+ \frac{(n-R_1-\dots-R_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1,r;m-1;n}^{(R_1, \dots, R_{r-2}, R_{r-1}+R_r+1, \dots, R_m)^{(i+2,j)}} \right. \\ &\left. + \left\{ \frac{i+2}{i+1} \right\} \mu_{r-1,r;m-1;n}^{(R_1, \dots, R_{r-2}, R_{r-1}+R_r+1, \dots, R_m)^{(i+1,j)}} \right]. \end{aligned} \quad (2.22)$$

Proof. (2.22) can be proved by putting $s = r + 1$ in (2.18).

Remark 2.1: Aggarawala and Balakrishnan (1996) have mentioned in Remark 5 pp. 762-763, that if $R_1 = R_2 = \dots = R_{j-1} = 0$, that is, there is no censoring before the j -th failure, then the first j progressively Type-II right censored order statistics are simply first j usual order statistics. Thus, for the special case $R_1 = R_2 = \dots = R_m = 0$ so that $m = n$, in which the progressively censored order statistics becomes the usual order statistics $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, whose single moments are denoted by $\mu_{r,n}^{(i)}$, $1 \leq r \leq n$ and product moments are denoted by $\mu_{r,s;n}^{(i,j)}$, $1 \leq r < s \leq n$, the recurrence relations established above would reduce to the following

(i) From (2.7): For $i \geq 0$,

$$\mu_{1:n}^{(i+2)} = \frac{(i+2)}{n\lambda} \left[\frac{(1+\lambda)}{\lambda} \mu_{1:n}^{(i)} + \left\{ 1 - \frac{n\lambda}{(i+1)} \right\} \mu_{1:n}^{(i+1)} \right].$$

(ii) From (2.14): For $2 \leq r \leq n-1$ and $i \geq 0$,

$$\begin{aligned} \mu_{r,n}^{(i+2)} &= \frac{(i+2)}{\lambda(n-r+1)} \left[\frac{(1+\lambda)}{\lambda} \mu_{r,n}^{(i)} + \left\{ 1 - \frac{\lambda(n-r+1)}{(i+1)} \right\} \mu_{r,n}^{(i+1)} \right] \\ &+ \left[\mu_{r-1,n}^{(i+2)} + \left\{ \frac{i+2}{i+1} \right\} \mu_{r-1,n}^{(i+1)} \right]. \end{aligned}$$

(iii) From (2.15): For $i \geq 0$,

$$\begin{aligned} & \mu_{n,n}^{(i+2)} \\ &= \frac{(i+2)}{\lambda} \left[\frac{(1+\lambda)}{\lambda} \mu_{n,n}^{(i)} + \left\{ 1 - \frac{\lambda}{(i+1)} \right\} \mu_{n,n}^{(i+1)} \right] + \left[\mu_{n-1,n}^{(i+2)} + \left\{ \frac{i+2}{i+1} \right\} \mu_{n-1,n}^{(i+1)} \right]. \end{aligned}$$

(1v) From (2.10): For $1 \leq r < s \leq n$ and $i, j \geq 0$,

$$\begin{aligned} & \mu_{r,s;n}^{(i,j+2)} \\ &= \frac{(j+2)}{\lambda(n-s+1)} \left[\frac{(1+\lambda)}{\lambda} \mu_{r,s;n}^{(i,j)} + \left\{ 1 - \frac{\lambda(n-s+1)}{(j+1)} \right\} \mu_{r,s;n}^{(i,j+1)} \right] \\ &+ \left[\mu_{r,s-1;n}^{(i,j+2)} + \left\{ \frac{j+2}{j+1} \right\} \mu_{r,s-1;n}^{(i,j+1)} \right]. \end{aligned}$$

(v) From (2.17): For $1 \leq r \leq n-1$ and $i, j \geq 0$,

$$\begin{aligned} \mu_{r,n;n}^{(i,j+2)} &= \frac{(j+2)}{\lambda} \left[\frac{(1+\lambda)}{\lambda} \mu_{r,n;n}^{(i,j)} + \left\{ 1 - \frac{\lambda}{(j+1)} \right\} \mu_{r,n;n}^{(i,j+1)} \right] \\ &+ \left[\mu_{r,n-1;n}^{(i,j+2)} + \left\{ \frac{j+2}{j+1} \right\} \mu_{r,n-1;n}^{(i,j+1)} \right]. \end{aligned}$$

Similarly, the remaining recurrence relations for product moments of progressive Type-II right censored order statistics can be reduce for the recurrence relation for the product moments of order statistics.

Remark 2.2: Let us assume that $\mu_{1:n}$ is known. Setting $i = 0$ in (2.8), we can get $\mu_{1:1:n}^{(n-1)(2)}$ by the knowledge of $\mu_{1:1:n}^{(n-1)} = \mu_{1:n}$. And for $i=1$, $\mu_{1:1:n}^{(n-1)(3)}$ can be evaluated from $\mu_{1:1:n}^{(n-1)(1)}$ and $\mu_{1:1:n}^{(n-1)(2)}$. From (2.7), $i=0$ and $m=2$, by the knowledge of $\mu_{1:2:n}^{(R_1, R_2)(1)}$, one can evaluate $\mu_{1:2:n}^{(R_1, R_2)(2)}$. And, for $i=1$, $\mu_{1:2:n}^{(R_1, R_2)(3)}$ can be evaluated from $\mu_{1:2:n}^{(R_1, R_2)(1)}$ and $\mu_{1:2:n}^{(R_1, R_2)(2)}$. Similarly setting $i=0$ and $m=3$, by the knowledge $\mu_{1:3:n}^{(R_1, R_2, R_3)(1)}$, one can evaluate $\mu_{1:3:n}^{(R_1, R_2, R_3)(2)}$. Therefore to evaluate the value of $\mu_{1:m;n}^{(R_1, R_2, \dots, R_m)(i+2)}$, we require values of $\mu_{1:m;n}^{(R_1, R_2, \dots, R_m)(1)}$ for all m . Now, setting $i=0$ in (2.14), $\mu_{2:3:n}^{(R_1, R_2, R_3)(2)}$ can be evaluated by the knowledge of $\mu_{2:3:n}^{(R_1, R_2, R_3)(1)}$. Thus for the recursive computation we have to find out $\mu_{r:m;n}^{(R_1, R_2, \dots, R_m)(1)}$ for all r, m , by integration and then remaining by the recursive manner.

Setting $i=0$ and $m=2$ in (2.9), we can get $\mu_{1,2:2;n}^{(R_1, R_2)(2,j)}$ by the knowledge of $\mu_{1,2:2;n}^{(R_1, R_2)(1,j)}$. Hence to evaluate $\mu_{1,2:m;n}^{(R_1, R_2, \dots, R_m)(i,j)}$, we need $\mu_{1,2:m;n}^{(R_1, R_2, \dots, R_m)(1,j)}$ or $\mu_{1,2:m;n}^{(R_1, R_2, \dots, R_m)(i,1)}$. By using (2.10), one can easily evaluate $\mu_{r,r+1;m;n}^{(R_1, R_2, \dots, R_m)(i,j)}$ for all i, j by the knowledge of $\mu_{1,2;m;n}^{(R_1, R_2, \dots, R_m)(i,1)}$.

Conclusions

Above investigations demonstrated that using the above relations, we can obtain all the single as well as

product moments of all progressively Type-II right censored order statistics for all sample sizes and all censoring schemes in a simple recursive way. Since recurrence relations reduce the amount of direct computation and hence the time and labour. Therefore the relations under consideration may be useful in computing the moments of any order of progressively Type-II right censored order statistics from Lindley distribution.

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